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Interpreting Inertial Fusion Neutron Spectra

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A burning laser fusion plasma produces a neutron spectrum first described by Brysk[1]. This and more recent work deals with the spectrum produced by a single fluid element. The distribution of temperatures and velocities in multiple fluid elements combine in any real spectrum; we derive formulas for how the neutron spectrum averages these contributions. The single element momentum spectrum is accurately Gaussian, but the multi-element spectrum exhibits higher moments. In particular, the skew and kurtosis are likely to be large enough to measure. Even the single fluid element spectrum may exhibit measurable directional anisotropy, so that instruments with different lines of sight should see different yields, mean velocities, mean temperatures, and higher moments. Finally, we briefly discuss how scattering in the imploded core modifies the neutron spectrum by changing the relative weighting of fuel regions with different temperatures and velocities.

I. INTRODUCTION

Arguably the most important diagnostic of conditions in the burning core of an inertial fusion implosion is the spectrum of DT neutrons. We interpret neutron peak width as the mean ion temperature of the burn, and the energy shift of the peak as the mean fluid velocity of the burning fuel. In this work we address the question, "How much information beyond temperature and mean velocity we might be able to extract from finer details of the peak shape?" Features of the neutron spectrum besides the DT peak at 14 MeV and the DD peak at 2.45 MeV are also useful (for example, the number of down-scattered neutrons is diagnostic of the areal density of cold fuel), but here we only discuss peak shape.

At the National Ignition Facility (NIF), we field an array of nTOF (neutron time of flight) spectrometers[2–5] and one MRS (magnetic recoil spectrometer)[6], sampling the neutron spectrum along very different lines of sight. The precision of these NIF neutron spectrometers is better than the approximations Brysk[1] makes in his 1972 paper. Ballabio[7] and Appelbe[8, 9] have provided more accurate formulas. Beyond those corrections, we sometimes observe significant variations in spectral shape with line of sight, as well as deviations from the Gaussian peak shape described by Brysk. We will derive formulas for interpreting such observations in terms of inhomogeneities, fluid velocity anisotropies, and correlations between fluid velocity and temperature in the burning fuel.

The shape of a neutron spectral peak is nearly a copy of the distribution of the center of mass (CM) velocities of the reacting ion pairs, or more precisely, the component of the CM velocity along the detector line of sight. Ultimately, therefore, the CM velocity distribution is the most we can learn from the peak shape. This CM velocity distribution carries no information about ordinary space, so neutron peak spectroscopy cannot measure anything related to position. In particular, neither shear, nor rotation, nor turbulence[10] (eddy scales) has any unique signature in the CM velocity distribution. More importantly, density in velocity space does not distinguish the thermal and bulk motion contributions to the CM velocity. Thus, peak width is only an “effective” ion temperature indicator, which includes both mean temperature and variance in fluid velocity component along the line of sight. With multiple lines of sight, we can distinguish the anisotropic part of the fluid velocity distribution, but the mean temperature and isotropic part of the velocity variance remain inextricably mixed.

The distribution of temperature and fluid velocity, that is, the number of neutrons produced in fluid elements in each bin in 4D (T, \mathbf{u}) space is very broad for imploded cores, typically spanning at least a factor of two in temperature and hundreds of km/s in fluid velocity. (“Stagnation” is a misnomer in real implosions.) These distributions dominate the detailed features of the peak shape. We develop relations between moments of the spectral peak and moments of the underlying (T, \mathbf{u}) distribution, which reduce to the formulas for single fluid element spectral shape derived in previous work.

Scattering changes the spectrum by filtering the (T, \mathbf{u}) distribution from neutrons born to neutrons detected. This re-weighting of the contributions of the various fluid elements can be significant, especially for DD. The intuition that DD and DT neutrons come from the same fluid elements may be misleading when the DD neutrons we detect come from fluid elements closer to our detectors than the detected DT neutrons. Ideas to disentangle thermodynamic

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temperature from fluid velocity variance, such as by comparing the widths of the DT and DD peaks, will be seriously compromised by scattering.

In addition to the formulas for spectral peak shape, we also provide numerical values for the various constants and fits to the reaction cross sections that appear in our formulas.

II. NEUTRON SPECTRUM FROM A SINGLE FLUID ELEMENT

Consider first a single element of burning plasma, limited to, say, a micron in size and a picosecond in time. This fluid element has a single ion temperature T and a single fluid velocity \mathbf{u} . We want the neutron spectrum in any given direction $\mathbf{\Omega}$, that is, the fraction of neutrons produced with a given lab frame momentum \mathbf{p} , within some momentum volume $d^3\mathbf{p}$. In the following section, we will add up the spectra produced by all such elements.

We work in $c = k = 1$ units, so that mass, kinetic energy, and temperature all have energy units, and velocity is unit-less.

A. Single reaction kinematics

We begin with the kinematics of a single DT (DDn) reaction, $D+T \rightarrow {}^4\text{He}+n$ ($D+D \rightarrow {}^3\text{He}+n$). To zeroth order, the neutron flies off with kinetic energy K_0 equal to the mass difference between reactants and products, roughly 14 MeV (2.45 MeV), corresponding to a velocity of $c/6$ ($c/14$). This neutron speed is high enough that relativistic corrections are about 3% (0.5%).

In the CM frame of two particles, $E = M + K$ is the total energy, where $K = K_1 + K_2$ is the sum of kinetic energies of the two particles and $M = m_1 + m_2$ is the sum of their masses. In the CM frame, the individual kinetic energies are in the ratio of the particle masses, augmented by half the total kinetic energy:

$$(m_1 + K/2) K_1 = (m_2 + K/2) K_2.$$

Hence the fraction of the total kinetic energy and corresponding momentum belonging to particle 1 are:

$$K_1 = (m_2 + K/2) K/E, \tag{1}$$

$$\begin{aligned} p_1^2 &= (2m_1 + K_1) K_1 \\ &= (2M + K) K (m_1 + K/2)(m_2 + K/2)/E^2. \end{aligned} \tag{2}$$

On the product side of the reaction, the total mass-energy E is unchanged, but the mass defect Q between products and reactants means that the CM kinetic energy on the product side is $Q + K$, very much larger than the incident kinetic energy K . According to Eq.(1), the neutron gets

$$K_n = (m_{He} + (Q + K)/2) (Q + K)/E. \tag{3}$$

We have already defined the neutron energy $K_0 = K_n$ at $K = 0$; now we have the expression to compute it. We will write $E_n = m_n + K_n$ for the total reaction CM frame neutron energy, $E_0 = m_n + K_0$ for the total neutron energy at $K = 0$. We also define p_n and $v_n = p_n/E_n$ to be the CM neutron momentum and velocity, respectively, corresponding to K_n , with p_0 and v_0 as their $K = 0$ counterparts. The K (or E) derivatives of the neutron properties are:

$$dE_n/dK = dK_n/dK = 1 - E_n/E, \tag{4}$$

$$dv_n/dK = (1 - v_n^2)(1 - E_n/E)/p_n \tag{5}$$

The energy of a neutron born in a fusion plasma differs from K_0 by far more than Eq.(3) suggests. The thermal motion responsible for the relative kinetic energy K of the reactants also creates a CM velocity \mathbf{v} of any reactant pair, distributed as a Maxwellian with a variance T/M in each component. Boosting \mathbf{p}_n by velocity \mathbf{v} produces

$$K'_n = \gamma(K_n + \mathbf{v} \cdot \mathbf{p}_n) + (\gamma - 1)m_n. \tag{6}$$

The big energy shift from this boost is the $\mathbf{v} \cdot \mathbf{p}_n$ term in Eq.(6), which selects the component of \mathbf{p}_n along the boost direction. The fractional change in neutron kinetic energy from the boost is $vp_n/K_n \sim 2v/v_0$, while the fractional change from the relative kinetic energy Eq.(3) is only K/Q . For example, in a 5 keV plasma, $2v/v_0 \sim 0.012$, while a typical DT reaction happens at $\sim 5T$ where $K/Q \sim 0.0014$. Therefore, for the most part, a thermonuclear neutron

m_n (MeV)	939.565379(21)	
	D+T \rightarrow α +n	D+D \rightarrow ^3He +n
K_0 (MeV)	14.0283920(15)	2.4486857(13)
v_0 (km/s)	51233.5921(27)	21601.8587(58)
M (MeV)	4684.53386(11)	3751.225718(85)
$M^{-1/2}$ (km/s/keV $^{1/2}$)	138.512035(16)	154.786865(17)
$(1 - E_0/M)/p_0$ (km/s/keV)	1.465126376(84)	3.25390044(88)

Table I: Neutron spectrum parameters for the DT and DD reactions from CODATA 2010, with uncertainties in parentheses. $M^{-1/2}$ is the conversion from $T^{1/2}$ to standard deviation of velocity. $(1 - E_0/M)/p_0$ is the conversion from K to our scaled momentum ω of Eq.(12).

spectrum records the distribution of the component of CM velocity along the viewing direction. We will consider the shift due to relative kinetic energy K to be a small correction to the boost by the CM velocity \mathbf{v} .

The boost transform of neutron momentum and energy Eq.(6) occurs several times in our analysis. The three most important reference frames are the individual reaction CM (unprimed symbols), the fluid rest frame (primed symbols), and the lab frame (double primed). The boost transform from CM to fluid frame is

$$E'_n = \gamma(E_n + \mathbf{v} \cdot \mathbf{p}_n) \quad (7)$$

$$p'_{n\parallel} = \gamma(p_{n\parallel} + vE_n) \quad (8)$$

$$p'_{n\perp} = p_{n\perp},$$

where the \parallel -direction is parallel to \mathbf{v} , and \perp is either perpendicular direction. The case that the neutron motion \mathbf{p}_n is parallel to the boost velocity \mathbf{v} turns out to be particularly important; for this special case the familiar collinear relativistic velocity addition formula applies:

$$v'_n = (v_n + v)/(1 + v_n v) \approx v_n + (1 - v_n^2)v. \quad (9)$$

Note that the neutron velocity v_n is high enough that classical velocity addition $v_n + v$ is wrong by a few percent even when the boost velocity v is small.

Later, we will need to find the boost velocity \mathbf{v} which transforms a CM momentum \mathbf{p}_n to a particular momentum \mathbf{p}'_n in the boosted frame. The easiest way to derive this relationship is to subtract the forward transform Eq.(7) and Eq.(8) from its inverse (which is the same with minus signs and primes swapped), leading to

$$\gamma\mathbf{v}/(\gamma + 1) = (\mathbf{p}'_n - \mathbf{p}_n)/(E'_n + E_n). \quad (10)$$

Squaring produces an expression for the $\gamma - 1$ associated with the boost in terms of the neutron momenta in the two frames, which will become central to our analysis:

$$\gamma - 1 = \frac{(\mathbf{p}'_n - \mathbf{p}_n)^2}{E'_n E_n + m_n^2 + \mathbf{p}'_n \cdot \mathbf{p}_n}. \quad (11)$$

For numerical work, we present the single reaction values that will appear in our results in Table I.

When we look at a neutron spectral peak, we are (for the most part) reading the distribution of the component of CM velocity along our line of sight. We propose to take this quite literally and use velocity as the independent variable for our neutron spectra rather than energy. However, if we were to use neutron velocity directly, the $(1 - v_n^2)$ factor in Eq.(9) would require us to dilate our spectral scale slightly. To mitigate this effect of relativistic velocity addition, we will work with neutron momentum spectra, scaled to units of velocity by a factor of E_0 . Differences in this scaled momentum are a very good surrogate for the underlying reactant CM velocity differences. From Eq.(8), the scaled momentum difference $(p'_{n\parallel} - p_{n\parallel})/E_0$ much more nearly equals v than the actual velocity difference $v'_n - v_n$. We will show that the neutron momentum spectrum distorts the shape of the CM velocity distribution of reacting pairs by less than the neutron velocity spectrum, and by far less than the neutron energy spectrum.

Since the zero order effect of the boost is to shift the centroid of the CM velocity spectrum from zero (in the fluid rest frame) to roughly v_0 , we will also shift our velocity scale by v_0 , so that we will write neutron momentum spectra in terms of the shifted and scaled neutron momentum variable

$$\omega = p''_n/E_0 - v_0. \quad (12)$$

(Again, the double prime frame is the lab frame of our spectrometer.) We can think of ω as the part of the neutron velocity due to the line-of-sight component of the CM velocity of the reactants which produced it, to the extent that any quantity independent of K can represent that CM velocity.

B. Thermal distribution of reactants

We assume the reactant ions are an ideal gas, so that their density of states is proportional to the momentum-volume $d^3\mathbf{p}'_1$, and the occupation fraction is $\exp(-K'_1/T)$, where K'_1 is the kinetic energy corresponding to the momentum \mathbf{p}'_1 . We use a prime to distinguish the fluid rest frame from the CM frame of an individual reaction from previous section. For relativistic K'_1 , this is the Maxwell-Jüttner distribution, which captures the onset of relativistic deviations from the Maxwell-Boltzmann distribution.

No practical fusion plasma has high enough temperature for any significant relativistic deviations from Maxwell-Boltzmann statistics. However, Appelbe[9] has demonstrated that the algebra of Maxwell-Jüttner statistics is almost equally amenable to deriving analytic formulas. Eventually, we will expand in powers of v/c and v/v_0 and discard small terms, but by preserving a full relativistic treatment to the end, we can see exactly what we are throwing away.

Owing to the fact that volume, hence particle density, is not Lorentz invariant, the relativistic density of states is a much more subtle concept than in non-relativistic statistical mechanics. The problem is that all the particles with a particular momentum \mathbf{p}'_2 have their own rest frame, different from the rest frame of the fluid. Hence, their density in their own rest frame differs from their density in the fluid rest frame. It is the density in the fluid rest frame to which the Maxwell-Jüttner distribution applies:

$$dn'_2 = N_{J2} n'_2 d^3\mathbf{p}'_2 \exp(-K'_2/T), \quad (13)$$

where N_J is the normalization factor to make $\int dn'_2 = n'_2$. In the rest frame of the particles, which we denote by a star superscript, the density of particles with momentum \mathbf{p}'_2 is lower, namely $dn_2^* = dn'_2/\gamma_2^*$, where $\gamma_2^* = E'_2/m_2$ corresponds to velocity $v_2^* = p'_2/E'_2$.

How many reactions take place per unit volume per unit time at one momentum pair relative to any other? The answer is the ratio of reaction rates at the two momentum pairs. Unlike particle density, reaction rate per unit volume, or number of reactions per 4-volume, is Lorentz invariant, so it is the same in the rest frame of the subset of particles with a given velocity as in the fluid rest frame. Reaction cross sections are often tabulated as a function of the kinetic energy K_1^* of a projectile (particle 1) in the rest frame of a target (particle 2), so the rate is:

$$d\dot{n} = dn_1^* dn_2^* v_1^* \sigma(K_1^*) \quad (14)$$

(When 1 and 2 are identical, as in DD, this rate double counts reactions, so there is an additional factor of 1/2.) Here dn_1^* is density of projectiles 1 in the rest frame of the target particle 2. This is related to dn'_1 by the ratio of two gamma factors, E'_1/m_1 for the transformation from the rest frame of 1 to the fluid rest frame, and γ_1^* to get from the rest frame of 1 to the rest frame of 2. Hence

$$d\dot{n} = dn'_1 dn'_2 \frac{m_2 p_1^*}{E'_1 E'_2} \sigma(K_1^*), \quad (15)$$

where the momentum $p_1^* = m_1 \gamma_1^* v_1^*$ of particle 1 in the rest frame of 2 corresponds to the kinetic energy K_1^* (and we again omit a factor of 1/2 for DD).

By combining the Maxwell-Jüttner distributions of the reactants Eq.(13) and the reaction rate Eq.(15), we arrive at the relative probability of a reaction with any pair of reactant momenta \mathbf{p}'_1 and \mathbf{p}'_2 :

$$dP(\mathbf{p}'_1, \mathbf{p}'_2) \propto n'_1 n'_2 \frac{d^3\mathbf{p}'_1}{E'_1} \frac{d^3\mathbf{p}'_2}{E'_2} m_2 p_1^* \sigma(K_1^*) \exp(-(K'_1 + K'_2)/T), \quad (16)$$

where the constant of proportionality is $N_{J1} N_{J2}$.

In order to apply the lessons we learned from our analysis of the single reaction, we must rewrite this relative probability in terms of the CM velocity \mathbf{v} in the fluid rest frame (and its corresponding γ), and the total kinetic energy K in the CM frame. (We should write \mathbf{v}' and γ' to indicate they are in the fluid rest frame, but we omit the primes in this case. Note that $-\mathbf{v}$ is the velocity of the fluid rest frame in the CM frame.) A more convenient variable for many of our purposes is the spacelike component of the CM 4-velocity $(\gamma, \gamma\mathbf{v})$,

$$\mathbf{w} = \gamma\mathbf{v} = \gamma(\mathbf{p}_1 + \mathbf{p}_2)/(E_1 + E_2). \quad (17)$$

We begin with the Lorentz invariance of the combination

$$\frac{d^3\mathbf{p}}{E} = 2\delta(E^2 - \mathbf{p}^2 - m^2) dE d^3\mathbf{p}, \quad (18)$$

which applies to any 4-vector whose Lorentz-invariant m^2 remains constant while its space-like part \mathbf{p} varies. Thus,

$$\frac{d^3\mathbf{p}'_1}{E'_1} \frac{d^3\mathbf{p}'_2}{E'_2} = \frac{d^3\mathbf{p}_1}{E_1} \frac{d^3\mathbf{p}_2}{E_2} \quad (19)$$

In this expression, $d^3\mathbf{p}_j$ represents the differential volume element around the CM frame momentum \mathbf{p}_j ; it is not quite the CM frame throughout the volume element. In that spirit, we transform to sum and difference momentum coordinates in that CM frame:

$$\begin{aligned} \mathbf{p} &= \mathbf{p}_1 + \mathbf{p}_2 \\ \mathbf{p}_- &= (\mathbf{p}_1 - \mathbf{p}_2)/2 \end{aligned} \quad (20)$$

This transform has unit Jacobian. A fixed value of the CM frame momentum difference \mathbf{p}_- corresponds to a fixed CM total energy, in other words a fixed mass-energy E for the reactant pair. In the CM frame, when E is fixed small changes in velocity (to slightly out of the CM frame) are related to the small changes in total momentum by $d\mathbf{p} = E d\mathbf{v}_{cm}$. Therefore, in the CM frame, $d^3\mathbf{p} = E^3 d^3\mathbf{v}_{cm}$. For infinitesimal $d\mathbf{v}_{cm}$ have $\gamma_{cm} = 1$, so that $d^3\mathbf{v}_{cm} = d^3\mathbf{w}_{cm}/\gamma_{cm}$. Since Eq.(18) applies to the 4-velocity, we can transform from the CM frame back to the fluid frame with $d^3\mathbf{w}_{cm}/\gamma_{cm} = d^3\mathbf{w}/\gamma$, so that

$$\frac{d^3\mathbf{p}'_1}{E'_1} \frac{d^3\mathbf{p}'_2}{E'_2} = \frac{E^3}{E_1 E_2} \frac{d^3\mathbf{w}}{\gamma} d^3\mathbf{p}_-. \quad (21)$$

To express $d^3\mathbf{p}_- = p_- dp_- d\Omega$ in terms of K , we use $p_- dp_- = E_1 dK_1 = E_2 dK_2$ to write

$$p_- dp_- = \frac{E_1 E_2}{E} dK.$$

We must also express the momentum and kinetic energy of particle 1 in the rest frame of 2, p_1^* and K_1^* , in terms of the total relative kinetic energy in the CM frame. Consider the Lorentz invariant $E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2$, which is $m_2 E_1^*$ in the rest frame of 2, and $E_1 E_2 + p_-^2$ in the CM frame. In the CM frame, we also have the identity $(E^2 - M^2)/2 = E_1 E_2 + p_-^2 - m_1 m_2$, so that

$$m_2 K_1^* = (M + K/2)K. \quad (22)$$

For the momentum, a Lorentz transform from the CM frame to the rest frame of 2 has velocity $v_2 = -p_-/E_2$ with $\gamma_2 = E_2/m_2$, so that $p_1^* = \gamma_2 p_- (1 + E_1/E_2)$, or

$$m_2 p_1^* = E p_-.$$

Finally, since $E' = \gamma E$, we can rewrite the exponent in Eq.(16) to show its dependence on v (or γ) and K explicitly:

$$K'_1 + K'_2 = (\gamma - 1)M + \gamma K = (\gamma - 1)E + K$$

Putting all this together, we arrive at the relative probability for a reaction to occur with CM 4-velocity \mathbf{w} , total kinetic energy in the CM frame K , and direction Ω of the incident reactants in the CM frame (the direction of \mathbf{p}_-):

$$\begin{aligned} dP(\mathbf{w}, K, \Omega) &= dP_{cm}(\mathbf{w}, K) dP_{rel}(K, \Omega), \quad \text{where} \\ dP_{cm} &\propto (d^3\mathbf{w}/\gamma) \exp(-(\gamma - 1)E/T) \\ dP_{rel} &\propto dK d\Omega E^3 p_-^2 \sigma(K_1^*) \exp(-K/T). \end{aligned} \quad (23)$$

Eq.(22) expresses the energy at which the cross section σ is to be evaluated in terms of K , and the total energy $E = M + K$. Eq.(2) gives $p_- = p_1$ as a function of K . We factor this probability into a part dP_{rel} depending only on the relative velocity and a part that depends on the CM velocity with only a very weak dependence on the relative kinetic energy K . In the non-relativistic limit dP_{cm} is independent of K ; the only K -dependence comes from the very small $(\gamma - 1)K/T$ term in the exponent.

The normalization integral for dP_{cm} (and similar integrals we need later) is an integral representation of modified Bessel functions of the second kind[11], K_ν . For any practical temperature $T \ll E$, the asymptotic series for these

functions converges rapidly:

$$e^z K_\nu(z) = \frac{z^\nu}{(2\nu-1)!!} \int_1^\infty d\gamma (\gamma^2 - 1)^{\nu-1/2} \exp(-(\gamma-1)z), \quad (24)$$

$$= \sqrt{\frac{\pi}{2z}} \left[1 + \sum_{i=1}^\infty \frac{\prod_{j=1}^i (4\nu^2 - (2j-1)^2)}{i!(8z)^i} \right], \quad (25)$$

$$= \sqrt{\frac{\pi}{2z}} N_\nu(z). \quad (26)$$

(For $\nu = 0$, $(2\nu-1)!! = 1$.)

The normalization factor N_J for the the Maxwell-Jüttner distribution Eq.(13) reduces to Eq.(24) with $\nu = 2$, so that

$$N_J = (2\pi mT)^{3/2} N_2(m/T).$$

For our dP_{cm} distribution, $z = E/T$ is larger than 10^5 for any T less than 50 keV (40 keV for DD), so the normalization series $N_\nu(E/T)$ is very nearly unity for any ν . Since $\gamma d\gamma = wdw$, $d^3\mathbf{w}/\gamma = 4\pi(\gamma^2 - 1)^{1/2} d\gamma$, which is Eq.(24) with $\nu = 1$. Thus, the normalized \mathbf{w} distribution for a given K is

$$dP_{cm} = \frac{d^3\mathbf{w}}{\gamma} \frac{\exp(-(\gamma-1)E/T)}{(2\pi T/E)^{3/2} N_1(E/T)}. \quad (27)$$

The normalization factors add a second weak K dependence to that in the exponential through $E = M + K$.

The normalization factor N_J for Eq.(13) is Eq.(24) with $\nu = 2$:

$$N_J = (2\pi mT)^{3/2} N_2(m/T).$$

The complete integral of the right hand side of Eq.(16) is the total reaction rate \dot{n} , and the mean reactivity $\bar{\sigma v} = \dot{n}/(n'_1 n'_2)$ by definition. Combining these normalization conditions with Eq.(23) and Eq.(27), we arrive at the normalized probability distribution for dP_{rel} (integrated over $d\Omega$):

$$\begin{aligned} \frac{dP_{rel}}{dK} &= N_{rel} N_1\left(\frac{E}{T}\right) \left(\frac{E}{T}\right)^{3/2} p_-^2 \sigma(K_1^*) \exp\left(-\frac{K}{T}\right), \\ N_{rel} &= \frac{(2/\pi)^{1/2} (m_1 m_2)^{-3/2}}{N_2(m_1/T) N_2(m_2/T) \bar{\sigma v}(T)}. \end{aligned} \quad (28)$$

Eq.(2) gives the K dependence of $p_- = p_1$. The cross section $\sigma(K_1^*)$ is the cross section for particle 1 with energy K_1^* in the rest frame of particle 2, where Eq.(22) gives K_1^* in terms of K .

The integral of Eq.(28) is the relativistically correct formula for reactivity

$$\bar{\sigma v}(T) = N_{rel} \int dK N_1\left(\frac{E}{T}\right) \left(\frac{E}{T}\right)^{3/2} p_-^2 \sigma(K_1^*) \exp\left(-\frac{K}{T}\right). \quad (29)$$

Since the CM and relative motion are not perfectly independent, our split into CM and relative probability distributions Eq.(27) and Eq.(28) is somewhat arbitrary. We choose dP_{rel}/dK to be the integrand of the formula for $\bar{\sigma v}$. We present fits to $\bar{\sigma v}(T)$ in Appendix A. For any temperature less than an MeV, the relativistic corrections present in Eq.(29) are negligible.

C. Neutron spectrum

Eq.(27) is the distribution of CM 4-velocities of reacting pairs with a given K , and among those reacting pairs, K is distributed according to Eq.(28). We must now find the corresponding distribution of the momenta of the neutrons produced in those reactions. The K distribution depends on the empirical cross section $\sigma(K_1^*)$, which we must treat with numerical methods. Fortunately, we know that K plays only a minor role in the neutron spectrum, and the \mathbf{w} distribution is amenable to analytic treatment. Hence, we begin by fixing the value of K and deriving the neutron spectrum for the population of reactions having precisely that relative kinetic energy. We finish by integrating that analytic result over the distribution of K actually present.

We want the distribution of neutron momenta in the fluid frame, or the probability a reaction will produce a neutron in $d^3\mathbf{p}'_n$. At fixed \mathbf{p}_n or $(K, \mathbf{\Omega})$ in the CM frame, there is some CM 4-velocity \mathbf{w} which will boost it to any given \mathbf{p}'_n , given by Eq.(10). We need to work out the Jacobian $d^3\mathbf{w}/d^3\mathbf{p}'_n$ for that transformation. For directions \perp perpendicular to \mathbf{w} , w and γ are unchanged for small variations dw_\perp , so that Eq.(10) says

$$dw_\perp/(\gamma + 1) = dp'_{n\perp}/(E'_n + E_n).$$

For variations dw_\parallel parallel to \mathbf{w} , we can use Eq.(8) and Eq.(7) to find that

$$dp'_{n\parallel} = E'_n dw_\parallel / \gamma.$$

Multiplying these, we have

$$\frac{d^3\mathbf{w}}{\gamma} = \frac{d^3\mathbf{p}'_n}{E'_n} \left(\frac{\gamma + 1}{E'_n + E_n} \right)^2. \quad (30)$$

Squaring Eq.(10), we find

$$\frac{\gamma + 1}{E'_n + E_n} = \frac{E'_n + E_n}{E'_n E_n + m_n^2 + \mathbf{p}'_n \cdot \mathbf{p}_n}, \quad (31)$$

which will become important in a moment.

The change of variables from \mathbf{v} to \mathbf{p}'_n in Eq.(23) has transformed the distribution Eq.(27) into

$$dP_{cm} = \frac{d^3\mathbf{p}'_n}{E'_n} \left(\frac{\gamma + 1}{E'_n + E_n} \right)^2 \frac{\exp(-(\gamma - 1)E/T)}{(2\pi T/E)^{3/2} N_1(E/T)}, \quad (32)$$

The fluid frame momentum \mathbf{p}'_n distribution must eventually be isotropic, but for fixed \mathbf{p}_n , the \mathbf{p}'_n distribution will be very strongly peaked in the \mathbf{p}_n direction, since the CM velocity is tiny compared to the neutron velocity. The \mathbf{p}'_n distribution we seek is the average over all directions of \mathbf{p}_n . Now fixing \mathbf{p}'_n , the only dependence on the direction of \mathbf{p}_n is via the dot product in Eq.(31), which we write in terms of the cosine between the CM and fluid neutron directions μ , $\mathbf{p}'_n \cdot \mathbf{p}_n = p'_n p_n \mu$. From Eq.(31),

$$\frac{d\gamma}{d\mu} = -p'_n p_n \left(\frac{\gamma + 1}{E'_n + E_n} \right)^2. \quad (33)$$

The directional mean of the μ -dependent factors in Eq.(32) is therefore

$$\int \frac{d\mathbf{\Omega}_n}{4\pi} \left(\frac{\gamma + 1}{E'_n + E_n} \right)^2 \exp\left(-\frac{(\gamma - 1)E}{T}\right) = \frac{T}{2Ep'_n p_n} \exp\left(-\frac{(\gamma(\mu) - 1)E}{T}\right) \Big|_{\mu=-1}^{\mu=+1} \quad (34)$$

The $\mu = -1$ term in Eq.(34) represents the reactions whose CM velocity \mathbf{v} is larger than and opposite to the neutron velocity, so that the neutron is moving in the opposite direction in the fluid frame than in the CM frame. At any practical fluid temperature, there are no such reactions: The integral of the $\mu = -1$ term is smaller than the integral of the $\mu = +1$ term by more than 20 orders of magnitude for any temperature T less than 1.75 MeV for DT, or 250 keV for DD. That is, for a yield of 10^{20} neutrons, we make less than a single neutron error by dropping the $\mu = -1$ term, for any temperature less than these limits. For practical temperatures, the error is of course many orders of magnitude smaller still. For now we simply drop the $\mu = -1$ limit of integration, and understand γ to be evaluated at $\mu = +1$, that is, where \mathbf{p}'_n is parallel to \mathbf{p}_n .

Eq.(34) is a major simplification. The CM frame direction $\mathbf{\Omega}_n$ is very nearly the fluid frame direction $\mathbf{\Omega}'_n$, but some range of directions must contribute. Eq.(34) allows us to consider only the case that \mathbf{p}_n and \mathbf{p}'_n are exactly collinear (specifically, $\mu = +1$). The effective solid angle replacing the eliminated $d\mathbf{\Omega}_n$ is roughly $2\pi(T/M)/v_0^2$, representing the range of \mathbf{p}_n directions which actually contribute. Numerically, this works out to a little under 4 ppm of the sphere per keV for DT and 26 ppm/keV for DD.

Although \mathbf{w} directions contribute almost isotropically, with Eq.(34) we need only compute the case of \mathbf{w} collinear with \mathbf{p}'_n (according to Eq.(10)). The boost transform Eq.(7) is now elementary, and the simple collinear velocity addition formula Eq.(9) applies to the neutron velocities $v'_n = p'_n/E'_n$ and $v_n = p_n/E_n$. Eq.(11) becomes

$$\gamma - 1 = \frac{(p'_n - p_n)^2}{E'_n E_n + m_n^2 + p'_n p_n}. \quad (35)$$

The p'_n variable is the magnitude of \mathbf{p}'_n , and therefore $p'_n > 0$. However, if we formally extend the range of p'_n to $p'_n < 0$, we get back the dropped $\mu = -1$ term in the angular integral Eq.(34). That is, we can imagine that a given direction $\boldsymbol{\Omega}'_n$ includes neutrons moving in the opposite direction, represented by negative magnitude $p'_n < 0$. Integrals over $d\boldsymbol{\Omega}'_n$ now cover the sphere twice, corresponding to the $\mu = \pm 1$ terms. Since the $\mu = -1$ term is so minuscule, we will not carefully follow this extended lower integration limit, but it does restore “exact” status to our results, despite apparently dropping the $\mu = -1$ term after Eq.(34).

We make one final boost from the fluid rest frame to the lab frame, written with double primes, to get the spectrum for a single fluid element with temperature T and velocity \mathbf{u} , with corresponding γ_u . The time-like component of the boost is $E'_n = \gamma_u(E''_n - \mathbf{u} \cdot \mathbf{p}''_n)$. We rearrange this to show the relationship between the magnitudes of the momenta and energy in the two frames as

$$E'_n = \gamma_u(E''_n - u_{\parallel} p''_n) \quad (36)$$

$$(1 - u^2)p_n'^2 = (p''_n - u_{\parallel} E''_n)^2 + u_{\perp}^2 m_n^2, \quad (37)$$

where u_{\parallel} is the component of \mathbf{u} parallel to \mathbf{p}''_n , and u_{\perp} is the magnitude of the two perpendicular components. The Lorentz invariant Eq.(18) is

$$p_n''^2 dp_n'' d\boldsymbol{\Omega}''_n / E'_n = p_n'^2 dp_n' d\boldsymbol{\Omega}'_n / E'_n.$$

We arrive at the neutron spectrum of a single fluid element with temperature T and velocity \mathbf{u} , for a given value of relative kinetic energy K :

$$\frac{dP_{cm}(K)}{d\omega d\boldsymbol{\Omega}''_n} = \frac{(E/T)^{1/2}}{2(2\pi)^{3/2} N_1(E/T)} \frac{E_0 p_n''^2}{E''_n p'_n p_n} \exp\left(-(\gamma - 1)\frac{E}{T}\right), \quad (38)$$

where ω is our scaled and shifted lab momentum variable from Eq.(17). This is a normalized probability distribution; it integrates to one. To get the complete neutron spectrum, we must average this single- K spectrum over the distribution of K given by Eq.(28),

$$\frac{dP}{d\omega d\boldsymbol{\Omega}''_n} = \int dK \frac{dP_{rel}}{dK} \frac{dP_{cm}(K)}{d\omega d\boldsymbol{\Omega}''_n}. \quad (39)$$

We could write the neutron spectrum Eq.(38) as a function of neutron energy K''_n , instead of as a function of scaled momentum ω . In fact, this practice is nearly universal in the literature. The factor to convert Eq.(38) from a momentum spectrum to an energy spectrum is

$$E_0 d\omega / dK''_n = E''_n / p''_n = 1/v''_n.$$

However, for the purpose of learning about a fusion plasma, this is a mistake. Again, the neutron spectrum is (mostly) a boosted replica of one component of the CM velocity distribution of the reactant pairs, so we really want to look at a velocity spectrum. As we discussed near the end of section II A, the best way to do this is to scale the momentum spectrum by E_0 when we want velocity units. The momentum spectrum is far more nearly Maxwellian than the energy spectrum, since it is essentially a boosted version of one component of the thermal momentum distribution.

We have written Eq.(38) to show clearly how relativistic effects are small corrections to the non-relativistic treatment, and to avoid high order cancellations which make many otherwise correct relativistic results difficult to accurately compute. To recap, $E = M + K$ where $M = m_1 + m_2$, while $E_0 = m_n + K_0$ where K_0 is the neutron kinetic energy at $K = 0$ from Eq.(3). Eq.(37) gives p'_n , $N_1(z)$ is from Eq.(26), and Eq.(3) gives K_n , hence p_n as a function of K , and Eq.(35) gives $\gamma - 1$. dP_{rel}/dK comes from Eq.(28).

With the conversion to a neutron energy spectrum and for a fluid element at rest ($\mathbf{u} = 0$), Eq.(39) agrees exactly with Eq.(36) of Appelbe[9], albeit written and derived very differently. (Appelbe makes a trivial mistake in his Eq.(33), resulting in an error by a factor of 4π in his Eq.(36).) When \mathbf{u} is non-zero, Appelbe’s Appendix B discussion omits the overall factor of p''_n/p'_n (for an energy spectrum) produced by the boost to non-zero fluid velocity \mathbf{u} , which our Eq.(39) includes. The various other single fluid element results in the literature, such as Brysk[1] or Ballabio[7], agree with Eq.(39) up to some accuracy, typically missing terms of order v^3/v_n^3 and v^2 .

D. Moments of the single element spectrum

We turn now to the ω moments of Eq.(38). We will use spectral moments heavily in our analysis of multiple fluid elements. We define the m th moment of the neutron spectrum Eq.(38) for a single K as

$$\mathcal{M}_m(T, \mathbf{u}; K) = 4\pi \int \frac{dp_n''}{E_0} \frac{dP_{cm}(K)}{d\omega d\boldsymbol{\Omega}''_n} \omega^m. \quad (40)$$

For these moment calculations, we will switch back to $dp_n'' = E_0 d\omega$. We choose the normalization constant so that the integral of \mathcal{M}_0 over $d^3\mathbf{p}_n''$ is 4π , that is, we normalize \mathcal{M}_0 per sphere instead of per steradian. Thus, the integral of \mathcal{M}_0 over p_n'' is a function of viewing direction $\boldsymbol{\Omega}_n''$ for non-zero \mathbf{u} . Note that \mathcal{M}_m has units of m th power of velocity.

Because the zeroth moment must integrate to 4π over all viewing directions for arbitrary \mathbf{u} , the $m = 0$ integrand must be a perfect differential in direction. Defining direction cosine $\mu = u_{\parallel}/u$ (the part of \mathbf{u} parallel to \mathbf{p}_n'' , not related to the previous μ), we find from Eq.(36) that at fixed E_n' and u ,

$$\frac{d(\mu p_n'')}{d\mu} = \frac{v_n'' p_n''}{v_n'' - u_{\parallel}}.$$

On the other hand, for fixed μ , the differential of Eq.(36), combined with the differential of Eq.(7) (when \mathbf{p}_n and \mathbf{v} are parallel) is, with $w = \gamma v$ from Eq.(17),

$$\frac{dp_n''}{p_n} = \frac{dw}{\gamma_u} \left(1 + \frac{v}{v_n}\right) \frac{1}{v_n'' - u_{\parallel}}.$$

The $m = 0$ integrand excluding the exponential factor, written as a μ derivative for fixed v_n' and u , is therefore

$$\frac{dp_n''}{E_n''} \frac{p_n'^2}{p_n' p_n} = \frac{dw}{\gamma_u} \left(1 + \frac{v}{v_n}\right) \frac{d}{d\mu} \left(\mu \frac{p_n''}{p_n'}\right). \quad (41)$$

At $\mu = \pm 1$, $u_{\parallel} = \pm u$, and $p_n''/p_n' = \gamma_u(1 \pm u/v_n')$. Thus, the integral over all \mathbf{u} directions of the $d/d\mu$ factor is simply $4\pi\gamma_u$, so Eq.(41) verifying our assertion that the integral of \mathcal{M}_0 over all directions is one, for any \mathbf{u} .

As Eq.(41) hints, we can simplify all the moment integrals by changing the integration variable from p_n'' to w . First we solve Eq.(37) for $p_n'' - u_{\parallel} E_n''$ and divide by Eq.(36) to produce a “pseudo-velocity” ψ_n' depending only on v_n' and \mathbf{u} :

$$\psi_n' \equiv \gamma_u \left((1 - u_{\parallel}^2) v_n'^2 - u_{\perp}^2 \right)^{1/2} = \frac{v_n'' - u_{\parallel}}{1 - u_{\parallel} v_n''}. \quad (42)$$

We recognize this as simple velocity addition, so $v_n'' = (\psi_n' + u_{\parallel})/(1 + u_{\parallel} \psi_n')$. Simple velocity addition Eq.(9) also relates v_n' , hence ψ_n' , to v_n and $v \ll v_n$. From Eq.(36), we get E_n'/E_n'' in terms of v_n'' , leading to $E_n'/E_n'' = \gamma_u(1 + u_{\parallel}^2)/(1 + u_{\parallel} \psi_n')$. Hence,

$$p_n''/p_n' = \gamma_{\parallel} + (\gamma_{\parallel}^2/\gamma_u)(u_{\parallel}/v_n') - \Psi, \quad (43)$$

where we write $\gamma_{\parallel} = (1 - u_{\parallel}^2)^{-1/2}$ and

$$\Psi = \gamma_{\parallel} - \frac{\psi_n'}{\gamma_u(1 - u_{\parallel}^2)v_n'} = \gamma_{\parallel} \left[1 - (1 - \gamma_{\parallel}^2 u_{\perp}^2/v_n'^2)^{1/2} \right]. \quad (44)$$

We note that $\gamma_{\parallel} u_{\perp}/v_n'$ and $(\gamma_{\parallel}^2/\gamma_u)(u_{\parallel}/v_n')$ are at most a few percent.

Next we boost momentum from the CM frame to the fluid frame

$$p_n'/E_0 = \gamma(1 + v/v_n)p_n/E_0.$$

We combine this with Eq.(43) and the velocity addition formula Eq.(9) to arrive at

$$\omega = \frac{p_n}{E_0} \left\{ \gamma U_C - 1 + \frac{w}{v_n} U_V \right\} + \omega_{cm}, \quad (45)$$

where we define

$$U_V = \gamma_{\parallel} + (\gamma_{\parallel}^2/\gamma_u)v_n u_{\parallel} - \Psi \quad (46)$$

$$U_C = \gamma_{\parallel} + (\gamma_{\parallel}^2/\gamma_u)(u_{\parallel}/v_n) - \Psi \quad (47)$$

$$\omega_{cm} = (p_n - p_0)/E_0 \quad (48)$$

Using Eq.(43), we can also express Eq.(41) in terms of $U_{C,V}$:

$$\begin{aligned} \frac{dp_n''}{E_n''} \frac{p_n'^2}{p_n' p_n} &= \frac{dw}{\gamma_u} \left(U_C^* + \frac{v}{v_n} U_V^* \right), \text{ where } U_{C,V}^* = \frac{\partial(u_{\parallel} U_{C,V})}{\partial u_{\parallel}} \\ U_V^* &= \gamma_{\parallel}^2 \{ \gamma_{\parallel} + 2(\gamma_{\parallel}^2/\gamma_u) v_n u_{\parallel} - \Psi^* \} \\ U_C^* &= \gamma_{\parallel}^2 \{ \gamma_{\parallel} + 2(\gamma_{\parallel}^2/\gamma_u) (u_{\parallel}/v_n) - \Psi^* \} \\ \Psi^* &= \Psi - (\gamma_{\parallel}^2/\gamma_u^2) (1 - \gamma_{\parallel}^2 u_{\perp}^2/v_n'^2)^{-1/2} u_{\parallel}^2/v_n'^2 \end{aligned} \quad (49)$$

From Eq.(49) and Eq.(40), the integrand for the m -th moment is

$$dw \{ U_C^* \omega^m / \gamma_u + (v/v_n) U_V^* \omega^m / \gamma_u \} \exp(-(\gamma - 1)E/T). \quad (50)$$

The factor in curly braces in the moment integrand Eq.(50) mostly depends on w through the explicit w/v_n term in Eq.(45) and the explicit v/v_n term in Eq.(49). A much weaker w dependence comes from the $\gamma = (1 + w^2)^{1/2}$ in Eq.(45) and the v_n' dependence of Ψ and Ψ^* . We therefore expand γ , Ψ , and Ψ^* as power series in w . With powers of w , we recognize the moment integrals from Eq.(24). In fact, changing variables to w and integrating that result by parts yields two useful families of integrals (with $z = E/T$):

$$\mathcal{J}_{\nu}(z) = \int_{-\infty}^{\infty} dw v w^{2\nu-1} \exp(-(\gamma - 1)z) = (2\nu - 1)!! (2\pi/z)^{1/2} z^{-\nu} N_{\nu}(z), \quad (51)$$

$$\mathcal{I}_{\nu}(z) = \int_{-\infty}^{\infty} dw w^{2\nu} \exp(-(\gamma - 1)z) = (2\nu - 1)!! (2\pi/z)^{1/2} z^{-\nu} N_{\nu+1}(z). \quad (52)$$

The \mathcal{I}_{ν} match the U_C^* term in Eq.(50), while the \mathcal{J}_{ν} match the U_V^* term. The former picks out only even powers of w , while the latter picks out only odd powers of w . Specifically, integrating over dw replaces $w^{2\nu}$ in the power series for $U_C^* \omega^m$ by \mathcal{I}_{ν} , and replaces $v w^{2\nu-1}$ in the power series for $U_V^* \omega^m$ by \mathcal{J}_{ν} . Since $\mathcal{I}_0(E/T)$ is precisely the reciprocal of the normalization factor in Eq.(40), each \mathcal{M}_m becomes a sum of terms proportional to either $\mathcal{I}_{\nu}/\mathcal{I}_0$ or $\mathcal{J}_{\nu}/\mathcal{I}_0$, each of which is itself a power series in $1/z = T/E$.

If we additionally expand γ_{\parallel} and γ_u as power series in u_{\parallel} and u , respectively, we will have each \mathcal{M}_m as a power series in T , u_{\parallel} , and u . The coefficients in these series will be functions of v_n , E , and ω_{cm} , with an overall m th power of $p_n/E_0 = 1 + \omega_{cm}$.

To begin this program, we note that expanding $\gamma = (1 + w^2)^{1/2}$ in powers of w is straightforward. To expand the weak w dependence of Ψ , which arises via v_n' , we must first expand Ψ in powers of $\gamma_{\parallel} u_{\perp}/v_n'$:

$$\Psi = \frac{1}{2} \gamma_{\parallel}^3 u_{\perp}^2 / v_n'^2 \left(1 + \frac{1}{4} \gamma_{\parallel}^2 u_{\perp}^2 / v_n'^2 + \frac{1}{8} \gamma_{\parallel}^4 u_{\perp}^4 / v_n'^4 + \dots \right), \quad (53)$$

For each term of this series, we convert v_n' to v_n by means of the velocity addition formula Eq.(9), expanded as a series in w :

$$\frac{v_n}{v_n'} = 1 - \frac{(1 - v_n^2)w/v_n}{(1 + w^2)^{1/2} + w/v_n} = 1 - (1 - v_n^2)(w/v_n)(1 - w/v_n + \dots) \quad (54)$$

We use a computer algebra system to keep track of all the polynomial terms, up to whatever order in u and w we please. After the substitution for the integration, we keep whatever powers of $1/z = T/E$ we please. Formally, T/E and ω_{cm} are of order v^2/v_0^2 . We take u to be of order v , while v_n is of order 1. Although the series converge relatively quickly, with three independent variables characterizing the fluid element — T , u_{\parallel} , and u — plus our unspecified reaction collision energy K , which implicitly determines the v_n and E parameters, the equations for \mathcal{M}_m quickly become very lengthy. For example,

$$\begin{aligned} \frac{\mathcal{M}_1}{p_n/E_0} &= \left\{ \frac{\omega_{cm}}{p_n/E_0} - \frac{u^2}{2v_n^2} + \left(1 + \frac{v_n^2}{2} \right) \frac{T}{v_n^2 E} \right\} + \left\{ 1 + \frac{2\omega_{cm}}{p_n/E_0} - \frac{3 + 2v_n^2}{2} \frac{u^2}{v_n^2} - \frac{9T}{2E} \right\} \frac{u_{\parallel}}{v_n} \\ &+ \frac{5 + v_n^2}{2} \frac{u_{\parallel}^2}{v_n^2} + \frac{5 + 7v_n^2}{2} \frac{u_{\parallel}^3}{v_n^3} + \frac{5 + 66v_n^2 + 9v_n^4}{8} \frac{u_{\parallel}^4}{v_n^4} - \mathcal{O}(v^4/v_0^4) \end{aligned} \quad (55)$$

E. Allowing for the relative kinetic energy distribution

The spectrum Eq.(38) depends only weakly on K , so our strategy is to expand in K . We begin by examining the K distribution dP_{rel}/dK from Eq.(28) and Eq.(29). We can pre-compute not only $\overline{\sigma v}(T)$, for any cross section, but also the n th moment of the dP_{rel}/dK distribution:

$$\overline{K^n}(T) = \int dK K^n dP_{rel}/dK. \quad (56)$$

For $n = 1$, we define a dimensionless $\mathcal{K}_1(T) = \overline{K}/T$, which is roughly 5, since most of thermonuclear reactions occur in the tail of the relative velocity distribution of the reactants. For $n > 1$, we define the dimensionless n th central moment \mathcal{K}_n :

$$\overline{(K - \overline{K})^n}(T) = \int dK (K - \overline{K})^n dP_{rel}/dK = T^n \mathcal{K}_n(T). \quad (57)$$

We provide fits to $\mathcal{K}_1(T)$ and $\mathcal{K}_2(T)$ in Appendix A.

For $n \geq 2$, the $\mathcal{K}_n(T)$ are non-linear functions of the distribution dP_{rel}/dK . For our spectral moment analysis, we will need linear functions of dP_{rel}/dK , so we define the non-dimensional raw moments \mathcal{K}'_n as well:

$$\overline{K^n}(T) = \int dK K^n dP_{rel}/dK = T^n \mathcal{K}'_n(T). \quad (58)$$

$$\mathcal{K}'_n(T) = \sum_{m=0}^n (-1)^m \binom{n}{m} \overline{K^m} \mathcal{K}_m(T). \quad (59)$$

Let $f(K)$ be an arbitrary function of K , such as the spectrum Eq.(38) at a particular \mathbf{p}''_n , T , and \mathbf{u} , or one of the moments \mathcal{M}_m . We can expand f as a Taylor series around \overline{K} :

$$f(K) = f(\overline{K}) + f'(\overline{K})(K - \overline{K}) + \frac{1}{2}f''(\overline{K})(K - \overline{K})^2 + \dots, \quad (60)$$

which makes it easy to compute the mean of f over the distribution $i_0(K)$ in terms of our precomputed K -distribution moments:

$$\overline{f(K)} = f(\overline{K}) + \frac{1}{2}f''(\overline{K})T^2\mathcal{K}_2 + \frac{1}{6}f'''(\overline{K})T^3\mathcal{K}_3 + \dots \quad (61)$$

When f is the spectrum Eq.(38), the strongest K dependence comes from the $\gamma - 1 \sim w^2$ in the exponential. The first K derivative is

$$T \frac{\partial}{\partial K} \frac{dP(K)}{d\omega d\mathbf{\Omega}''_n} \bigg/ \frac{dP(K)}{d\omega d\mathbf{\Omega}''_n} = \mathcal{R} \left(\frac{w}{v_n} - \frac{T}{Ev_n^2} \right) + \frac{T}{2E} - (\gamma - 1) - \frac{N'_1}{N_1}, \quad (62)$$

where we define $\mathcal{R} = E/E_n - 1$. The w/v_n term in Eq.(62) is largest, of order v/v_0 ; the N'_1 term is of order v^4/v_0^4 , and all the rest are of order v^2/v_0^2 . Each successive K derivative multiplies the previous derivative by Eq.(62) in addition to differentiating it with respect to K , hence the n -th derivative will be of order v^n/v_0^n . We have multiplied the derivative in Eq.(62) by a factor of T to track the power of K in the expansion Eq.(61), so that each term in that expansion is smaller than the previous term by a factor of order v/v_0 , that is, the ratio of thermal to neutron velocity. The leading order term for the n -th derivative is

$$T^n \frac{\partial^n}{\partial K^n} \frac{dP(K)}{d\omega d\mathbf{\Omega}''_n} \bigg/ \frac{dP(K)}{d\omega d\mathbf{\Omega}''_n} = \mathcal{R}^n \left(\frac{T}{Ev_n^2} \right)^{n/2} \text{He}_n \left(\left(\frac{w^2 E}{T} \right)^{1/2} \right) + \dots, \quad (63)$$

where He_n are the Hermite polynomials.

With the first correction term, the single fluid element spectrum Eq.(39) becomes

$$\frac{dP}{d\omega d\mathbf{\Omega}''_n} = \left(1 + \mathcal{K}_2 \frac{w^2 E - T}{2Ev_n^2} \mathcal{R}^2 \right) \frac{dP_{cm}(\overline{K})}{d\omega d\mathbf{\Omega}''_n}. \quad (64)$$

The parameters depending upon K , namely v_n and E , are to be evaluated at $K = \bar{K}(T)$ throughout. This formula is accurate enough for many practical purposes, and is much cheaper to compute than the integral in Eq.(39). The \mathcal{K}_2 correction term broadens the peak slightly to account for the distribution of relative kinetic energies of the reactants. The overall shape is still quite accurately Gaussian for a single fluid element.

The moments of the single fluid element spectrum Eq.(39), or its approximate form Eq.(64) are

$$\mathcal{M}_m(T, \mathbf{u}) = \int dK \mathcal{M}_m(T, \mathbf{u}; K) dP_{rel}/dK. \quad (65)$$

We can apply the same formalism to the moments as to the whole spectrum. However, expanding around $K = 0$ for the moments will be more convenient than around $K = \bar{K}(T)$, since we want to use the moments in our analysis of spectra resulting from plasmas with a distribution of temperatures. The K dependence in the moments comes from $\omega_{cm} = p_n/E_0 - 1$, v_n , E , and an overall factor that is a power of $p_n = E_n v_n$. We have reduced each moment to a sum of terms which are powers of these quantities, so differentiating with respect to K is straightforward. The derivatives dv_n/dK from Eq.(5) and dE_n/dK from Eq.(4) introduce no new variables, and produce only sums of powers of v_n , E_n , and E , so we can take multiple K derivatives of the moments with no additional machinery.

The end result is a power series in K for each moment, which is a power series in T , $u_{||}$, and u , with coefficients that are functions of v_0 , E_0 , and M (the $K = 0$ values of v_n , E_n , and E). Integrating over the K distribution simply amounts to replacing each K^n by its mean \bar{K}^n . The moments simplify if we scale K to units of velocity, and T to units of velocity squared:

$$\tau = T/M \quad (66)$$

$$\kappa = \left(1 - \frac{E_0}{M}\right) \frac{K}{p_0} = K \left. \frac{d\omega_{cm}}{dK} \right|_0 \quad (67)$$

Note that κ is the shift in the centroid of the spectrum (from v_0) caused by the mean relative kinetic energy of the reactants at temperature T , to the extent that shift is proportional to K . We will present formulas for the moments in the next section. Averaging those formulas over K produces averages of powers of κ over K , which we write $\bar{\kappa}$, $\bar{\kappa}^2$, etc., proportional to the corresponding functions of temperature \mathcal{K}_1 , \mathcal{K}_2 , etc. defined in Eq.(58).

III. NEUTRON SPECTRUM FOR MULTIPLE FLUID ELEMENTS

In laser fusion implosions, burn occurs over a very wide range of different temperatures: Adiabatic temperature increase during compression turns burn on, then decreasing temperature shuts burn off. A typical NIF shot simulation produces significant neutrons over a range of a factor of 2 to 2.5 in temperature (from the 10% to the 90% point of the burn weighted temperature distribution). The standard deviation of the burn temperature distribution ranges from about 1/4 to 1/3 of its mean, with perfectly symmetric implosions on the lower end and highly distorted implosions on the upper end of that range. In 2D and 3D simulations with realistic asymmetry, significant burn likewise occurs over a very wide range of fluid velocities – say from 250 km/s away from the detector to 250 km/s toward the detector. (One dimensional implosion models exhibit a smaller range of fluid velocities during burn, but about the same range of burn temperatures.)

Usually our neutron spectrometer is far from the target, so that only the overall distribution of fluid temperatures and velocities from every point in the target and every time during the burn matters. Let this overall distribution function be $\mathcal{N}(T, \mathbf{u})$, so that the number of neutrons produced at any point of the four dimensional (T, \mathbf{u}) space, the “burn distribution” is

$$d\mathcal{N} = \mathcal{N}(T, \mathbf{u}) dT d^3\mathbf{u}. \quad (68)$$

For any function $f(T, \mathbf{u})$, we can define the burn average $\langle f \rangle$ as

$$\langle f \rangle = \int f d\mathcal{N} / \int d\mathcal{N}. \quad (69)$$

We can take f to be the entire spectrum from Eq.(64) or Eq.(39). Since the spectrum for a single fluid element Eq.(64) is very nearly a Gaussian with mean $u_{||}$ and variance T/E , the burn average spectrum is a sum of Gaussians. Even when all the $u_{||} = 0$, the sum is no longer Gaussian – it will have a sharper peak corresponding to the colder fluid elements and a broader tail due to the hotter fluid elements. In other words, a realistic neutron spectrum has kurtosis which increases with the variance of the burn temperature distribution. Any variance in the distribution of

u_{\parallel} , the velocity component along the viewing direction, will act to broaden the peak as well as to further distort it from a perfect Gaussian shape.

In addition to the mean $\langle \dots \rangle$ of a single function f over the burn distribution, we also define the generalized covariance of several functions f, g, h , and so on, as:

$$\text{Cov}(f, g, h, \dots) = \langle (f - \langle f \rangle)(g - \langle g \rangle)(h - \langle h \rangle) \dots \rangle \quad (70)$$

We will also abbreviate $\text{Var}(f) = \text{Cov}(f, f)$.

In order to quantify all of this, we take burn averages of the moments $\mathcal{M}_m(T, \mathbf{u})$. We have defined the \mathcal{M}_m to be linear functions of the spectrum $dP/d\omega d\mathbf{\Omega}_n''$, so that the moments of the burn averaged spectrum are the burn averaged moments $\langle \mathcal{M}_m \rangle$. We begin with \mathcal{M}_0 , which is the yield in direction $\mathbf{\Omega}_n''$, relative to what it would be if the neutrons were emitted isotropically in the lab frame:

$$\begin{aligned} \langle \mathcal{M}_0 \rangle = & 1 + \frac{2}{v_0} \langle u_{\parallel} \rangle + \frac{1 + v_0^2}{2v_0^2} \left(3\langle u_{\parallel}^2 \rangle - \langle u^2 \rangle \right) - \frac{2(1 - v_0^2)}{v_0^2} \langle \bar{\kappa} u_{\parallel} \rangle \\ & + \frac{2}{v_0^3} \left(2\langle u_{\parallel}^3 \rangle - v_0^2 \langle u^2 u_{\parallel} \rangle \right) + \mathcal{O}(v^4/v_0^4). \end{aligned} \quad (71)$$

Kinematic effects make the yield slightly anisotropic, dominated by an L=1 variation with viewing direction, of amplitude $2|\langle \mathbf{u} \rangle|/v_0$. The small $\langle \bar{\kappa} u_{\parallel} \rangle$ corrects for the fact that v_0 rather than v_n appears in the denominator of the $\langle u_{\parallel} \rangle$ term.

For the higher moments, after the burn averaging, we divide by the burn averaged yield to get the unit-normalized spectral moment:

$$\langle \omega^m \rangle = \langle \mathcal{M}_m \rangle / \langle \mathcal{M}_0 \rangle. \quad (72)$$

To do this, we expand the reciprocal of $\langle \mathcal{M}_0 \rangle$ as a power series, and multiply by our power series for $\langle \mathcal{M}_m \rangle$. The first normalized moment is

$$\begin{aligned} \langle \omega \rangle = & \langle u_{\parallel} \rangle + \langle \bar{\kappa} \rangle + \frac{2 + v_0^2}{v_0} \langle \tau \rangle + \frac{5\text{Var}(u_{\parallel}) - (1 - v_0^2)\langle u_{\parallel}^2 \rangle - \langle u^2 \rangle}{2v_0} \\ & + \frac{2}{v_0} \text{Cov}(\bar{\kappa}, u_{\parallel}) + v_0 \langle \bar{\kappa} u_{\parallel} \rangle - \frac{2 + v_0^2}{v_0^2} \langle \tau \rangle \langle u_{\parallel} \rangle + \frac{9}{2} \langle \tau u_{\parallel} \rangle \\ & + \frac{(5 + 7v_0^2)}{2v_0^2} \text{Cov}(u_{\parallel}, u_{\parallel}, u_{\parallel}) + \frac{1 + 8v_0^2}{v_0^2} \text{Var}(u_{\parallel}) \langle u_{\parallel} \rangle + \langle u_{\parallel} \rangle^3 \\ & - \frac{3 + 2v_0^2}{2v_0^2} \text{Cov}(u^2, u_{\parallel}) - \frac{1}{2} \langle u^2 \rangle \langle u_{\parallel} \rangle - \frac{\langle \bar{\kappa}^2 \rangle}{2v_0} + \mathcal{O}(v^4/v_0^4). \end{aligned} \quad (73)$$

The $\langle \bar{\kappa}^2 \rangle$ term is the largest of order v^4/v_0^4 , we omit other terms formally of the same order. The first two terms are the only ones likely to be large enough to detect in NIF experiments; at $T = 5$ keV the third term is about 5 km/s (15 km/s) for DT (DD). All terms which include a variance or covariance are zero for a single fluid element, since by definition there is only a single temperature (or K) in that case.

The variance $\text{Var}(\omega) = \langle \omega^2 \rangle - \langle \omega \rangle^2$ is the square of the width of the distribution. We expand and combine the power series for the first and second moments to find

$$\begin{aligned} \text{Var}(\omega) = & \langle \tau \rangle + \text{Var}(u_{\parallel}) + 2\text{Cov}(\bar{\kappa}, u_{\parallel}) + \frac{4 + 3v_0^2}{v_0} \text{Cov}(\tau, u_{\parallel}) + 2v_0 \langle \tau \rangle \langle u_{\parallel} \rangle \\ & + \frac{3 + v_0^2}{v_0} \text{Cov}(u_{\parallel}, u_{\parallel}, u_{\parallel}) + \frac{2 + 2v_0^2}{v_0} \text{Var}(u_{\parallel}) \langle u_{\parallel} \rangle - \frac{1}{v_0} \text{Cov}(u^2, u_{\parallel}) \\ & + \langle \bar{\kappa}^2 \rangle - \langle \bar{\kappa} \rangle^2 + \frac{2}{v_0} \text{Cov}(\bar{\kappa}, \tau) + \mathcal{O}(v^4/v_0^4). \end{aligned} \quad (74)$$

We show only the two largest fourth order terms. The first term, the mean temperature, is the original Brysk interpretation of the peak width. The second term, the variance of fluid velocity, introduces a dependence of peak width on viewing direction – in terms of spherical harmonics it is an L=2 variation. This variation of apparent temperature with direction can amount to as much as 20% in simulations. Note that it is the burn averaged temperature and velocity variance which determine the variance of the neutron peak, when the peak is not Gaussian due to inhomogeneous burn conditions. Higher moments do contribute at higher orders, but reduced by a factor of v/v_0 per order.

For the third moment and beyond, we are interested in the deviations from a Gaussian spectral shape. Hence, we convert the moments to cumulants. When we expand and combine the power series, we find the skew and kurtosis (the cumulants corresponding to the third and fourth moments) to be

$$\begin{aligned} \text{Skew}(\omega)\text{Var}(\omega)^{3/2} &= 3\text{Cov}(\tau, u_{\parallel}) + \text{Cov}(u_{\parallel}, u_{\parallel}, u_{\parallel}) + 3\text{Cov}(\bar{\kappa}, \tau) \\ &\quad + \frac{6 + 9v_0^2}{2v_0}\text{Var}(\tau) + 3v_0\langle\tau\rangle^2 + 3\text{Cov}(\bar{\kappa}, u_{\parallel}, u_{\parallel}) + \mathcal{O}(v^4/v_0^4). \end{aligned} \quad (75)$$

$$\begin{aligned} \text{Kurt}(\omega)\text{Var}(\omega)^2 &= 3\text{Var}(\tau) + 6\text{Cov}(\tau, u_{\parallel}, u_{\parallel}) + \text{Cov}(u_{\parallel}, u_{\parallel}, u_{\parallel}, u_{\parallel}) - 3\text{Var}(u_{\parallel})^2 \\ &\quad + 12\text{Cov}(\bar{\kappa}, \tau, u_{\parallel}) + \mathcal{O}(v^5/v_0^5) + 3\langle\tau\rangle^3 + \dots \end{aligned} \quad (76)$$

In Eq.(75), we show only the four largest fourth order terms. In Eq.(76), we show only the largest fifth order term, and we also show the largest term which is non-zero for a single fluid element, $3\langle\tau\rangle^3$, which is sixth order.

The largest terms in the skew and kurtosis for a single fluid element are, respectively, $+3v_0\tau^{1/2}$ and $+3\tau$. These represent the intrinsic deviation of a thermal neutron spectrum from a Gaussian shape, that is, what would be present even if the burning plasma were absolutely uniform. For DT (DD) at $T = 5$ keV, the intrinsic skew is 5.3×10^{-4} (2.5×10^{-4}), scaling as $T^{1/2}$ – exceedingly small for any fusion plasma. The intrinsic kurtosis at 5 keV is 3.1×10^{-3} for DT (3.5×10^{-3} for DD), scaling as T . Both these terms arise from relativistic corrections to the neutron and CM frame velocity addition. The largest non-relativistic skew and kurtosis terms are $2\tau^{3/2}/v_0^3$ (from order v^6 terms) and $-6\tau^2/v_0^4$ (from order v^8 terms), respectively.

The neutron spectrum will exhibit substantial kurtosis when there is a distribution of burn temperatures. The first term in Eq.(76) produces a kurtosis of roughly $3\text{Var}(T)/\langle T \rangle^2$. We have mentioned that simulations predict $\text{Var}(T)/\langle T \rangle^2$ between about 1/16 and 1/9. Thus, typical implosions should produce neutron spectra with 0.2 to 0.3 kurtosis, on the lower side for perfectly symmetric implosions, and on the higher side for highly distorted implosions. An implosion with a realistic distribution of burn temperatures does not produce a Gaussian neutron spectrum.

The spectral skew for an asymmetric implosion will likely be dominated by the $3\text{Cov}(\tau, u_{\parallel})$ term, which has an $L=1$ directional variation. However, even a perfectly symmetric implosion will exhibit a skewed neutron spectrum owing to the $3\text{Cov}(\bar{\kappa}, \tau)$ and $3\text{Var}(\tau)/v_0$ terms. Estimating $\bar{K} \approx 5T$, we find $\text{Cov}(\bar{\kappa}, \tau) \approx 20\text{Var}(\tau)/v_0$. This ranges between 1.25 and 2.2 times $\langle\tau\rangle^2/v_0$, so we expect the skew in the neutron momentum spectrum to be 4 to 7 times $\langle\tau\rangle^{1/2}/v_0$ owing to the distribution of burn temperatures, much larger than the intrinsic skew. Therefore, at 5 keV, the skew for a perfectly symmetric implosion should be at least 0.02 for DT and 0.06 for DD, which may be measurable. Temperature-velocity correlations in simulations of highly asymmetric implosions produce much larger skew, in the range of 0.1–0.2 for DT.

We chose our non-standard spectral variable ω , the neutron momentum scaled to velocity units, at least partly because that choice minimizes distortions to the underlying CM Maxwellian. We can now back up that claim by comparing the intrinsic momentum spectrum skew to the intrinsic skew of the velocity or energy spectrum. The velocity spectrum skew is $-6v_0\tau^{1/2}$, that is, the opposite sign and twice the magnitude of the momentum spectrum skew. The energy spectrum skew is $+3\tau^{1/2}/v_0$, larger than the momentum spectrum skew by a factor of $1/v_0^2$, which is 34 for DT (193 for DD). Thus, energy is a poor choice for the spectral variable because it substantially distorts the underlying CM Maxwellian distribution we want to study: At 5 keV the intrinsic skew of the energy spectrum is 0.02 for DT (0.05 for DD), compared to 0.0005 (0.00025) for our scaled momentum spectrum.

IV. HOW SCATTERING CHANGES PEAK SHAPE

Out-scatter and small angle down-scatter can both change the shape of the peak significantly. Out-scatter operates in two different ways: First, the energy dependence of the total neutron scattering cross section causes differential removal within the peak. Second, correlations between position and fluid state (temperature and velocity) change the distribution $\mathcal{N}(T, \mathbf{u})$. That is, \mathcal{N} should be the distribution of neutrons detected in a particular line of sight, not the distribution of neutrons born moving in the direction.

Table II shows the total neutron out-scatter cross sections for three fuel compositions and three ablator materials. Typical fuel areal density in NIF cryogenic layered implosions is about 1 g/cm^2 , while ablator areal density at burn time is less than 0.3 g/cm^2 . Hence, roughly 20% of the DT neutrons and 45% of the DD neutrons scatter in high performance NIF implosions. Scattering modifies the DD peak considerably more than the DT peak.

	σ_{tot} at DT (cm ² /g)	σ_{tot} at DD (cm ² /g)
DT	0.215	0.556
DD	0.242	0.707
T ₃ H	0.219	0.564
Be	0.102	0.157
CH	0.092	0.193
C	0.066	0.080

Table II: Total neutron cross sections at the DT peak energy (14.028 MeV) and at the DD peak energy (2.449 MeV), for three fuel compositions and for three ablator materials. The cross sections are from ENDF/B-VII.1, or JENDL-4.0 for C.

A. Energy dependent out-scatter

We estimate the impact of energy dependent out-scatter on the peak shape by looking at how it would affect a purely Gaussian neutron momentum spectrum of mean $\bar{\omega}$ and variance σ_{ω} ,

$$dP/d\omega \propto \exp\left(-\frac{1}{2}(\omega - \bar{\omega})^2/\sigma_{\omega}^2\right) \exp(-\rho r \sigma_{tot}(\omega)). \quad (77)$$

Here $\sigma_{tot}(\omega)$ is the total neutron cross section showing its presumed energy dependence, and ρr is an average areal density for the scattering process. The actual process is far more complex, of course, but here we are merely trying to estimate the magnitude of the effect. Expanding the total cross section as a Taylor series in ω and discarding everything beyond ω^2 , we see that differential scattering across the peak simply shifts its mean and variance by

$$\bar{\omega} \rightarrow \left(\bar{\omega} - \sqrt{\text{Var}(\omega)} \Delta'_{\sigma} \tau_{tot}\right) / (1 + \Delta''_{\sigma} \tau_{tot}) \quad (78)$$

$$\text{Var}(\omega) \rightarrow \text{Var}(\omega) / (1 + \Delta''_{\sigma} \tau_{tot}), \text{ where} \quad (79)$$

$$\Delta'_{\sigma} = \sqrt{\text{Var}(\omega)} (d\sigma_{tot}/d\omega) / \sigma_{tot}, \quad (80)$$

$$\Delta''_{\sigma} = \text{Var}(\omega) (d^2\sigma_{tot}/d\omega^2) / \sigma_{tot}, \quad (81)$$

and $\tau_{tot} = \rho r \sigma_{tot}(0)$ is the scattering depth at the center energy of the peak. The Δ'_{σ} and Δ''_{σ} are fractional changes in cross section over one standard deviation of the peak. The cross section curvature term $\Delta''_{\sigma} \tau_{tot}$ is the fractional change in variance. Assuming that is relatively small, the cross section slope term $\Delta'_{\sigma} \tau_{tot}$ is the centroid shift as a fraction of its standard deviation $\sqrt{\text{Var}(\omega)}$.

For 14 MeV DT neutrons and various fuel compositions, the first derivative $M^{-1/2} \sigma_{tot}^{-1} d\sigma_{tot}/d\omega$ ranges from -0.004 to -0.005 keV^{-1/2} (T₃H has the highest slope, then DT, then DD). Hence the peak shift is about 1% of the peak width at $T=4$ keV (to higher energy) per unit optical depth. The second derivative $M^{-1/2} \sigma_{tot}^{-1} d^2\sigma_{tot}/d\omega^2$ is always small, below about 10⁻⁴ keV⁻¹, so the variance changes by less than 0.1% per unit optical depth for any temperature less than 10 keV. For 2.45 MeV DD neutrons, the first derivative ranges from +0.002 to -0.006 keV^{-1/2} (the tritium cross section increases with energy at 2.45 MeV), while the second derivative is once again below about 10⁻⁴ keV⁻¹. Again, this means DD peak shifts near 1% of peak width per unit optical depth at 4 keV temperature, with changes in variance below 0.1% per unit optical depth up to 10 keV. For DT, optical depth is typically near 0.2 for NIF high performance implosions, so both the peak shift and width change will be at most parts per thousand. For DD, the optical depth is typically around 0.5, which still leaves peak shifts below 1% of the peak width for NIF implosions.

Ablator materials like carbon or beryllium, on the other hand, have more structure in their cross sections σ_{tot} , so second order expansion in ω is a poor approximation. Nevertheless, we can compute the actual shifts and width changes for a variety of Gaussian centroids and variances to bound the magnitude of the effect. For 14 MeV DT neutrons and NIF-like temperatures in the range from 2-6 keV, peak shift ranges from -0.3% to +0.3% of the peak width per keV^{1/2} per unit optical depth for carbon or CH, and is about -0.2% for beryllium. Since ablator ρr should be considerably less than fuel ρr , in the range of 0.1-0.2 g/cm², the DT peak shift again ought to be no more than 1% of the peak width in NIF implosions. For 2.45 MeV DD neutrons, the shift as a fraction of peak width per keV^{1/2} per unit optical depth ranges from -0.5% to +0.4% for carbon or CH, but is much larger, about +5%, for beryllium. Thus, with a beryllium ablator at 0.2 g/cm² and a 4 keV burn temperature, out-scattering from the ablator would shift the DD peak to lower energy by less than 0.2% of the peak width, which is the biggest peak shift due to ablator scattering. The second derivative change in variance is also small, up to a maximum of 1% per keV per unit optical depth for beryllium and DD neutrons, but three to ten times smaller for DT neutrons or carbon or CH ablators.

Thus, the energy dependence of the total neutron scattering cross section will cause peak shifts of under 1% of peak width, and even smaller changes in variance for NIF layered implosions.

	DT neutrons		DD neutrons	
	σ_{el}^* (cm ² /g)	ω_{el} (km/s)	σ_{el}^* (cm ² /g)	ω_{el} (km/s)
DT	0.171	5320	0.304	3070
DD	0.175	7200	0.514	4400
T ₃ H	0.175	4980	0.260	2800
Be	0.068	900	0.101	1090
CH	0.031	530	0.075	1600
C	0.032	510	0.047	1210

Table III: Differential neutron cross section parameters for near forward scatter at the DT peak energy (14.028 MeV) and at the DD peak energy (2.449 MeV), for three fuel compositions and for three ablator materials.

B. Near forward scattering

Small angle elastic scattering returns some neutrons into the peak at a lower energy than before the scatter event. Usually, we estimate this small angle down-scatter by observing larger angle down-scatter outside the peak, so we can correct the peak shape. The peak shape change due to small angle down-scatter is the error in this correction procedure, for practical purposes.

Down-scatter into the peak is a more serious problem than energy dependence of out-scatter. To make some rough estimates of the problem, we note that for small angle scatter, the differential cross section is approximately exponential (or linear) as a function of energy loss. On our ω neutron momentum scale, we write the differential elastic cross section as

$$d\sigma_{el}/d\omega \approx \sigma_{el}^*/\omega_{el} \exp(\omega/\omega_{el}), \quad (82)$$

for $\omega < 0$, and $d\sigma_{el}/d\omega = 0$ for $\omega > 0$. Here, ω_{el} is the scale length in scaled neutron momentum corresponding to the forward elastic scatter peak. Table III shows our estimates of these forward scattering parameters from ENDL-2009 differential cross section tables.

Without any attempt to correct for near forward down-scatter, the errors in the raw spectral moments will be

$$\Delta\langle\omega^n\rangle = n!\omega_{el}^n\tau_{el}^*/(1 - \tau_{tot}), \quad (83)$$

where $\tau_{el}^* = \sigma_{el}^*\rho r$ is the optical depth for near-forward elastic scatter. For scattering off fuel in a layered NIF capsule with $\rho r \sim 1$ g/cm², this is of order 1000 km/s for the centroid shift, with correspondingly large errors in higher moments. Some kind of correction for near forward scatter from the fuel is therefore an absolute necessity. Since the problem is that ω_{el} is very large for scattering from the fuel, even the very simple correction of restricting the moment integral to within a few widths of the peak will reduce the centroid and variance errors by a large factor. For scattering off a CH or carbon ablator with a fairly large 0.3 g/cm² areal density, the near forward scatter error in centroid would be only of order 5 km/s, but the error in variance for completely ignoring near forward scatter from the ablator would be a substantial $2 \times 500^2 \times 0.01$ km²/s² or about 0.25 keV.

In this paper we merely point out that near-forward elastic scatter, both from fuel and ablator, will cause substantial distortion of the DT peak shape, as well as of the DD peak shape. The accuracy of any measurement of the peak centroid, variance, or higher moments will depend on precisely what technique we use to correct for near-forward scatter. A discussion of particular correction techniques is beyond our scope here.

C. Scattering changes to fluid temperature and velocity distribution

Perhaps the subtlest effect of scattering is to alter the distribution $\mathcal{N}(T, \mathbf{u})$ of fluid temperature and velocity introduced in section III. Only the neutrons actually detected contribute to this distribution, and the fraction of neutrons born in any given fluid element which reach the detector will vary with line of sight, as well as with position and time. For a given line of sight and at a given instant of time, each fluid element lies at some optical depth $\tau_{tot} = \int dr \rho \sigma_{tot}$ through the capsule to the detector. We define attenuation $\alpha = 1 - \exp(-\tau_{tot})$, the fraction of neutrons born in a given fluid element that are removed by scattering or absorption from a given line of sight. Writing $\mathcal{N}_0(T, \mathbf{u})$ for the distribution of neutrons born, we have for the distribution of detected neutrons

$$\mathcal{N}(T, \mathbf{u}) = (1 - \alpha)\mathcal{N}_0(T, \mathbf{u}). \quad (84)$$

We write $\langle \dots \rangle$ for mean values weighted by \mathcal{N} (neutrons detected), and $\langle \dots \rangle_0$ for mean values weighted by \mathcal{N}_0 (neutrons born), and similar 0 subscripts for $\text{Var}(\dots)$ and $\text{Cov}(\dots)$. For any quantity X ,

$$\langle X \rangle = \langle X \rangle_0 - \text{Cov}_0(\alpha, X) / (1 - \langle \alpha \rangle_0), \quad (85)$$

$$\text{Var}(X) = \text{Var}_0(X) - \left(\frac{\text{Cov}_0(\alpha, X)}{1 - \langle \alpha \rangle_0} \right)^2 - \frac{\text{Cov}_0(\alpha, X, X)}{1 - \langle \alpha \rangle_0}, \quad (86)$$

with more complicated expressions for higher cumulants. Such formulas tell us how to convert distributions from "as born" to "as detected". While the birth distributions are completely independent of the positions of the fluid elements, the detected distributions require knowledge about the relative location of emitter, scatterer or absorber, and detector. Specifically, each moment of X depends on covariances of X with α . The removal fraction α serves as a position coordinate, running from zero at the detector to some positive value, different for each ray, on the far side of the target. Out-scattering changes the mean values and variances of temperature and fluid velocity to the extent the fluid conditions are correlated with position.

When α is small, as in low ρr implosions, the corrections in Eq.(85) and Eq.(86) are small. However, in NIF layered implosions, the difference between detected and birth distributions can be substantial, especially for the DD peak, where α is roughly one half. The strongest effect is on the mean and variance of the component of velocity toward the detector. During the implosion, fuel on the far side of the capsule is moving toward the detector, while fluid on the near side is moving away from the detector. Since α is effectively a position coordinate, it can be highly correlated with the line of sight velocity u , with $\text{Cov}_0(\alpha, u_{\parallel}) > 0$ for an implosion. At late times when the capsule is exploding, on the other hand, $\text{Cov}_0(\alpha, u_{\parallel}) < 0$.

In non-igniting NIF implosions, burn mostly occurs during the implosion phase, so that the centroid of the neutron peak will be shifted to lower energy by this mechanism, according to Eq.(85). The square of this shift, which is the second term of Eq.(86), can cause a substantial reduction in the velocity variance as well. These reductions in both the energy of the peak and its width will appear even in completely symmetric implosions.

V. SUMMARY

We have extended the work of Brysk, Appelbe, and others, to describe the shape of the peak of a neutron spectrum for two-product thermonuclear fusion reactions like DT and DD. Unlike previous work, we include the effects of a distribution of fluid elements with many different temperatures and velocities. We have also recast previous single fluid element results in terms of the neutron momentum spectrum. The momentum spectrum offers a less distorted picture of the underlying CM velocity distribution than the velocity spectrum, and a much less distorted picture than the neutron energy spectrum used in all previous studies[1, 7–10].

We derive the momentum spectrum cumulants up to kurtosis, which show how successive moments of fluid temperature and velocity, weighted by number of neutrons produced in that fluid state, contribute to shape of the peak in the momentum spectrum. The dominant term in the kurtosis, for example, turns out to be three times the variance of the temperature, divided by the square of the mean temperature. For the range of temperatures of the burn in laser fusion capsules, the kurtosis of the momentum spectrum should be of order 0.25, which may be measurable with NIF neutron spectrometers. Since most laser fusion capsules, even exploding pushers at much smaller scales than NIF capsules, burn over a similarly broad temperature range, kurtosis should be a universal feature of neutron spectral peaks.

The only assumption we make here is that the reactant momentum distribution is Maxwell-Jüttner, that is, Maxwellian corrected for relativistic effects. However, neutron scattering in the capsule can substantially alter the spectrum, which may significantly complicate the relation between the moments of the spectrum and the moments of the underlying fluid temperature and velocity. In NIF cryogenic layered implosions, scattering removes roughly half of the DD neutrons, making any interpretation of DD spectra particularly challenging. Scattering will also complicate the variation of peak shape with viewing direction predicted by our moment formulas.

When kinetic effects are important, the reactant momentum distribution is not Maxwellian, and our formulas may break down. Albright[12] has discussed this case. We believe that layered NIF implosions are so collisional that even at 5σ in the tail of the relative velocity distribution where the fusion reactions take place, the reactant distribution is Maxwellian. We note that as long as the central part (not the tails) of the CM velocity distribution remains Maxwellian, our analysis applies. The tails of the relative velocity distribution affect reaction rates, hence $\mathcal{N}(T, \mathbf{u})$, but do not directly affect the neutron spectrum.

Reactant species separation has also been invoked to explain anomalies in fusion neutron spectra[13]. Again, we do not believe this is a significant effect for NIF layered implosions. Species separation by itself changes only the distribution of fluid temperatures and velocities $\mathcal{N}(T, \mathbf{u})$ at which neutrons are produced. If species separation does

take place, our formulas for spectral shape remain unchanged, as long as we use the $\mathcal{N}(T, \mathbf{u})$ describing where neutrons were actually born.

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Appendix A: Relative Kinetic Energy Distribution Moments

Bosch and Hale[14] describe the standard techniques for calculating and measuring the DT and DDn reaction cross sections $\sigma(K_1^*)$. We have performed similar fits to the 2011 ENDF/B-VII.1 cross sections. The Bosch and Hale fit to the DT reactivity is within a few tenths of a percent of our Eq.(29) with $\sigma(K_1^*)$ from the ENDF/B-VII.1 table, interpolated according to ENDF recommendations for the use of the table. This is about the level of errors we expect due to interpolation between points in the table, and also the accuracy claimed by Bosch and Hale. However, the Bosch and Hale fit to the DDn reactivity is not nearly as accurate; presumably the ENDF/B-VII.1 DDn table has changed since they published their fit.

Charged particle cross sections behave according to the Gamov asymptotic form at low energies, which gives the scaling due to tunneling through the Coulomb barrier. A steepest descents approximation to the reactivity integral Eq.(29) produces the asymptotic form for low temperature:

$$\bar{\sigma v}(T) \propto \xi^2 \exp(-3\xi), \quad \text{where} \quad (\text{A1})$$

$$\begin{aligned} \xi &= (T_G/T)^{1/3} \\ T_G &= \frac{\pi^2 m_{12}}{2} \left(\frac{e^2}{\hbar c} \right)^2. \end{aligned} \quad (\text{A2})$$

Bosch and Hale use a parameterization suggested by Peres[15]:

$$\begin{aligned} \bar{\sigma v}(T) &= C_1 \theta T^{-3/2} (\xi'/m_{12})^{1/2} \exp(-3\xi'), \\ \xi' &= (T_G/\theta)^{1/3}, \\ \frac{T}{\theta} &= 1 - \frac{C_2 T + C_4 T^2 + C_6 T^3}{1 + C_3 T + C_5 T^2 + C_7 T^3}. \end{aligned} \quad (\text{A3})$$

For the DDn reactivity, we find the following parameterization works much better than the Peres form:

$$\bar{\sigma v}(T) = P_1 \left(\frac{1 + P_2 T^{1/3} + P_3 T^{2/3} + P_4 T}{1 + P_5 T^{1/3} + P_6 T^{2/3}} \right) \xi^2 \exp(-3\xi). \quad (\text{A4})$$

This is a purely empirical fit; we have no theoretical justification for using a rational function in $T^{1/3}$. Unfortunately, this parameterization does not fit the DT reactivity as well as the Peres form.

In addition to the ENDF/B-VII.1 cross section tables, we have also used the ENDL-2011 cross section tables, which we use in radhydro simulations of laser fusion capsules. The ENDL and ENDF cross sections differ by a couple percent. Table IV shows our best Peres parameters for the DT reactivity, for use in Eq.(A3). The ENDF fit differs only slightly from the one given by Bosch and Hale. Table V shows our best Eq.(A4) fit parameters for the DDn reactivity.

We turn next to the central moments of the kinetic energy distribution, Eq.(57). As Brysk[1] discussed, in the non-relativistic limit,

$$T^n \mathcal{K}_n(T) = (-1)^n \frac{d^n \log(T^{3/2} \bar{\sigma v})}{d(1/T)^n}. \quad (\text{A5})$$

We use Eq.(A5) to derive the asymptotic form for the $\mathcal{K}_n(T)$. For the first two moments, the results are:

$$\begin{aligned} \mathcal{K}_1(T) &\rightarrow \xi + 5/6 \\ \mathcal{K}_2(T) &\rightarrow 2\xi/3 + 5/6 \end{aligned}$$

m_{12}	1124647 keV	
T_G	295.540 keV	
	ENDF/B VII.1	ENDL 2011
C_1	1.173×10^{-9}	1.173×10^{-9}
C_2	1.455×10^{-2}	1.166×10^{-2}
C_3	7.383×10^{-2}	6.094×10^{-2}
C_4	4.666×10^{-3}	4.661×10^{-3}
C_5	1.357×10^{-2}	1.352×10^{-2}
C_6	-1.076×10^{-4}	-1.134×10^{-4}
C_7	1.304×10^{-5}	1.377×10^{-5}

Table IV: Eq.(A3) parameters for DT reactivity, where T is in keV and $\overline{\sigma v}$ is in cm^3/s . The fits are accurate to better than 0.25% from 0.4 to 100 keV, and better than 0.4% from 0.1 to 500 keV.

m_{12}	937806 keV	
T_G	246.441 keV	
	ENDF/B VII.1	ENDL 2011
P_1	3.400×10^{-16}	3.453×10^{-16}
P_2	4.204×10^{-1}	1.783×10^{-1}
P_3	-2.120×10^{-2}	7.794×10^{-2}
P_4	5.959×10^{-2}	1.758×10^{-2}
P_5	2.945×10^{-1}	2.123×10^{-1}
P_6	4.671×10^{-2}	7.369×10^{-3}

Table V: Eq.(A4) parameters for DDn reactivity, where T is in keV and $\overline{\sigma v}$ is in cm^3/s . The fits are accurate to better than 0.1% from 0.1 to 500 keV.

However, taking derivatives of our reactivity fits loses accuracy, because the fits have a minimax character in which they oscillate around the true reactivity. Direct fits to the moments are substantially more accurate, as well as easier to compute. Replacing the 5/6 in the asymptotic forms by a simple rational function of T produces an economical parameterization for the $\mathcal{K}_n(T)$:

$$\mathcal{K}_1(T) = \xi + \mathcal{F}_1(T) \quad (\text{A6})$$

$$\mathcal{K}_2(T) = 2\xi/3 + \mathcal{F}_2(T) \quad (\text{A7})$$

$$\mathcal{F}_n(T) = \frac{N_0 + N_1T + N_2T^2 + N_3T^3 + N_4T^4}{1 + D_1T + D_2T^2 + D_3T^3 + D_4T^4} \quad (\text{A8})$$

We select the parameters for a minimax fit to $\mathcal{K}_n(T)$ calculated with the the full dP_{rel}/dK from Eq.(29), using the tabular cross sections $\sigma(K_1^*)$ from either ENDF or ENDL. For all fits except the DT $\mathcal{K}_2(T)$, $N_4 = D_4 = 0$ produces a fit accurate to a few tenths of a percent up to temperatures well over 100 keV, which is comparable to interpolation errors in the tables.

	ENDF/B VII.1	ENDL 2011
N_0	8.2771×10^{-1}	8.1857×10^{-1}
N_1	1.4316×10^{-1}	1.4272×10^{-1}
N_2	-4.4142×10^{-3}	-4.8217×10^{-3}
N_3	9.4336×10^{-6}	1.1536×10^{-5}
D_1	6.7860×10^{-4}	-2.7543×10^{-3}
D_2	9.5200×10^{-3}	9.2859×10^{-3}
D_3	2.6693×10^{-6}	5.1419×10^{-6}

Table VI: Eq.(A8) parameters for DT $\mathcal{K}_1(T)$, where T is in keV. The fits are accurate to better than 0.3% for ENDF and 0.4% for ENDL below 500 keV.

	ENDF/B VII.1	ENDL 2011
N_0	8.2430×10^{-1}	8.1838×10^{-1}
N_1	1.6705×10^{-1}	1.3628×10^{-1}
N_2	-1.8109×10^{-2}	-1.7565×10^{-2}
N_3	-8.0725×10^{-6}	7.0575×10^{-5}
N_4	3.6811×10^{-7}	7.4668×10^{-8}
D_1	-9.8389×10^{-2}	-1.2737×10^{-1}
D_2	2.0905×10^{-2}	2.2125×10^{-2}
D_3	3.2518×10^{-4}	1.0949×10^{-4}
D_4	-1.2273×10^{-7}	2.1816×10^{-8}

Table VII: Eq.(A8) parameters for DT $\mathcal{K}_2(T)$, where T is in keV. The ENDF fit is accurate to better than 0.3% below 300 keV; the ENDL fit to better than 0.4% below 130 keV.

	ENDF/B VII.1	ENDL 2011
N_0	8.4722×10^{-1}	8.4600×10^{-1}
N_1	9.9475×10^{-3}	9.2054×10^{-2}
N_2	-1.9279×10^{-3}	2.9147×10^{-4}
N_3	4.2125×10^{-5}	2.1762×10^{-6}
D_1	-1.0892×10^{-2}	7.7368×10^{-2}
D_2	-1.1420×10^{-3}	1.9492×10^{-4}
D_3	3.1340×10^{-5}	1.6537×10^{-6}

Table VIII: Eq.(A6) parameters for DD $\mathcal{K}_1(T)$, where T is in keV. The fits are accurate to better than 0.1% below 500 keV.

	ENDF/B VII.1	ENDL 2011
N_0	8.4793×10^{-1}	8.4557×10^{-1}
N_1	2.1394×10^{-1}	1.6161×10^{-1}
N_2	1.1918×10^{-2}	-3.8010×10^{-4}
N_3	4.5548×10^{-4}	5.5231×10^{-6}
D_1	2.0441×10^{-1}	1.2545×10^{-1}
D_2	8.9585×10^{-3}	-3.6591×10^{-4}
D_3	3.3944×10^{-4}	4.2665×10^{-6}

Table IX: Eq.(A7) parameters for DD $\mathcal{K}_2(T)$, where T is in keV. The ENDF fit is accurate to better than 0.4% below 450 keV; the ENDL fit to better than 0.1% below 500 keV.

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